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SUPERSOLUBLE CROSSED PRODUCT CRITERION FOR DIVISION ALGEBRAS

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ABSTRACT

Let D be a finite-dimensional F-central division algebra. A criterion is given for D to be a supersoluble (nilpotent) crossed product division algebra in terms of subgroups of the multiplicative group D^* of D. More precisely, it is shown that D is a supersoluble (nilpotent) crossed product if and only if D^* contains an abelian-by-supersoluble (abelian-by-nilpotent) generating subgroup.

1. Introduction

Let D be a division algebra with center F and degree n (i.e., $\dim_F D = n^2$). The algebra D is called **crossed product** if it contains a maximal subfield K such that K/F is Galois. D is said to be a **supersoluble** crossed product if $Gal(K/F)$ is supersoluble. We also recall that a subgroup G of D^* is a **generating** subgroup if $F[G] = D$. When $n = p$, a prime, it is shown in [1] that D is cyclic if and only if D^* contains a nonabelian soluble subgroup. Here we generalize this result to a division algebra of arbitrary degree n. To be more precise, it is proved that D is a supersoluble crossed product if and

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only if D^* contains an abelian-by-supersoluble generating subgroup. We then present a criterion for D to be a nilpotent (respectively, abelian or cyclic) crossed product. In fact, it is shown that a noncommutative finite-dimensional F-central division algebra D is a nilpotent (respectively, abelian or cyclic) crossed product if and only if there exist a generating subgroup G of D^* and an abelian normal subgroup A of G such that G/A is nilpotent (respectively, abelian or cyclic). We recall that soluble subgroups of the multiplicative group of a division ring were first studied by Suprunenko in [4].

2. Notations and conventions

Let D be a division ring with center F and G be a subgroup of D^*. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated by elements of G over F. We shall say that G is a **generating** subgroup if $D = F[G]$. For any group G we denote its center by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the **normalizer** of H in G, and $\langle H, K\rangle$ the group generated by H and K, where K is a subgroup of G. We shall say that H is **abelian-by-finite (abelian-by-supersoluble)** if there is an abelian normal subgroup K of H such that H/K is finite (supersoluble). Let S be a subset of D; then the **centralizer** of S in D is denoted by $C_D(S)$. For notations and results used in the text on central simple algebras, see [3].

3. Supersoluble crossed product division algebras

This section deals with a few results on division algebras whose multiplicative groups contain certain subgroups. Using these results we eventually prove our main theorem, which asserts that a division algebra D is a supersoluble crossed product if and only if D^* contains an abelian-by-supersoluble generating subgroup. Further criteria are also given for when a division algebra is a nilpotent (abelian or cyclic) crossed product. To be more precise, it is shown that a noncommutative finite-dimensional F-central division algebra D is a nilpotent (respectively, abelian or cyclic) crossed product if and only if there exist a generating subgroup G of D^* and an abelian normal subgroup A of G such that G/A is nilpotent (respectively, abelian or cyclic). We begin our study with the following:

LEMMA 3.1: *Let D be a finite-dimensional F-central division algebra. If D is a crossed product, then D^* contains an abelian-by-finite generating subgroup.*

Proof: Let K be a maximal subfield of D such that K/F is Galois. By the Skolem–Noether Theorem, for any $\sigma \in Gal(K/F)$ there exists an element $x \in N = N_{D^*}(K^*)$ such that $\sigma(k) = xkx^{-1}$, for all $k \in K$. Hence $N_{D^*}(K^*)/C_{D^*}(K^*) \simeq Gal(K/F)$. Since K is a maximal subfield of D, we have $C_{D^*}(K^*) = K^*$. Therefore, $N_{D^*}(K^*)$ is an abelian-by-finite subgroup of D^*. To complete the proof, it is enough to show that N is a generating subgroup, i.e., $F[N] = D$. This follows from the fact that by taking $x_\sigma \in N$ inducing σ, the x_σ form a basis of D over K. ∎

LEMMA 3.2: *Let D be a finite-dimensional F-central division algebra. Suppose that K is a subfield of D containing F. If G is a generating subgroup of D^* such that $K^* \lhd G$, then K/F is Galois and $G/C_G(K^*) \simeq Gal(K/F)$.*

Proof: Consider the homomorphism $\sigma: G \longrightarrow Gal(K/F)$ given by $\sigma(x) = f_x$, where $f_x(k) = xkx^{-1}$, for any $k \in K$. It is clear that $\ker \sigma = C_G(K^*)$. Now, we claim that $Fix(\operatorname{im}\sigma) = F$. Choose an element $a \in Fix(\operatorname{im}\sigma)$. For any $x \in G$ we have $f_x(a) = a$, and hence $xa = ax$. This shows that $a \in C_G(K^*) = F$ since G is a generating subgroup. So $Fix(Gal(K/F)) \subseteq Fix(\operatorname{im}\sigma) = F$, which implies that K/F is a Galois extension and σ is surjective. Therefore, we have $G/C_G(K^*) \simeq Gal(K/F)$ and K/F is a Galois extension. ∎

LEMMA 3.3: *Let D be a finite-dimensional F-central division algebra and let G be a generating subgroup of D^*. If K is a subfield of D containing F such that $[G : C_G(K^*)] = [K : F]$, then $C_D(K) = F[C_G(K^*)]$.*

Proof: Put $D_1 = C_D(K)$ and $D_2 = F[C_G(K^*)]$. It is clear that $D_2 \subseteq D_1$. By Corollary 7.1.4 of [2], $[D_1 : K] = [D : F]/[K : F]^2$, hence $[D : D_1] = [K : F]$. On the other hand, any element of D can be written in the form $\sum_{i=1}^{s} f_i g_i$, where $g_i \in G$ and $f_i \in F$, for any $1 \le i \le s$. Now, let $\ell = [G : C_G(K^*)]$ and $G = \bigcup_{i=1}^{\ell} C_G(K^*)x_i$. Therefore, every element of D can be written in the form $\sum_{i=1}^{\ell} a_i x_i$, where $a_i \in D_2$, for any $1 \le i \le \ell$. So $[D : D_2] \le \ell$ and we obtain $[D : D_1] \le [D : D_2] \le [G : C_G(K^*)] = [K : F] = [D : D_1]$, and so $D_1 = D_2$, as desired. ∎

Remark 1: Suppose that G is an (abstract) group which is either nilpotent or supersoluble. Let N be a nontrivial normal subgroup of G. We claim that there exists a nontrivial cyclic subgroup of N which is normal in G. To see this, we observe that if G is nilpotent, since $N \cap Z(G)$ is nontrivial we have nothing to prove. Now, suppose that G is supersoluble. Assume that $\langle e \rangle = N_t \subseteq \cdots \subseteq N_1 \subseteq N_0 = G$ is a normal series for G with cyclic factors. We observe that there exists a minimal natural number s such that $N \cap N_s$ is nontrivial and so $N \cap N_{s+1} = \langle e \rangle$. Therefore, we conclude that $N \cap N_s$ as a subgroup of N_s/N_{s+1} is cyclic, as required.

The following theorem provides a criterion for an F-central division algebra to be a supersoluble crossed product:

THEOREM 3.4: *Let D be a noncommutative finite-dimensional F-central division algebra. Then D is a supersoluble (respectively, nilpotent, abelian, or cyclic) crossed product if and only if there exist a generating subgroup G of D^* and an abelian normal subgroup A of G such that G/A is supersoluble (respectively, nilpotent, abelian, or cyclic).*

Proof: The "only if" part is clear from the proof of Lemma 3.1. Suppose that G is a generating subgroup of D^* and A is an abelian normal subgroup of G such that G/A is supersoluble (respectively, nilpotent, abelian, or cyclic). Take A maximal. Therefore, we have a maximal abelian normal subgroup A of G such that G/A is supersoluble (respectively, nilpotent, abelian, or cyclic). Set $G_1 = K^*G$, where $K = F(A)$. One may easily show that G_1 is a generating subgroup and K^* is a maximal normal abelian subgroup of G_1 such that $G_1/K^* \simeq G/A$. By Lemma 3.2, we conclude that K/F is Galois and $G_1/C_{G_1}(K^*) \simeq Gal(K/F)$. Therefore, K/F is supersoluble (respectively, nilpotent, abelian, or cyclic) Galois. To complete the proof, it is enough to show that K is a maximal subfield of D. First we claim that $C_{G_1}(K^*) = K^*$. Set $N = C_{G_1}(K^*)$ and suppose $K^* \subsetneq N$. It is easily seen that $N \lhd G_1$. We know that G_1/K^* is either supersoluble or nilpotent, so according to Remark 1, N/K^* contains a nontrivial cyclic subgroup which is normal in G_1/K^*. Thus, there exists $x \in C_{G_1}(K^*) \backslash K^*$ such that $\langle K^*, x \rangle / K^*$ is a normal subgroup of G_1/K^*, and hence $\langle K^*, x \rangle \neq K^*$ is an abelian normal subgroup of G_1. This contradicts the maximality of K^* in G_1. Therefore, $C_{G_1}(K^*) = K^*$ and the claim is established. Now, by Lemma 3.3, we obtain $F[C_{G_1}(K^*)] = C_D(K)$, and hence $C_D(K) = K$. Thus, K is a maximal subfield of D and the proof is complete. ∎

COROLLARY 3.5: *Let D be a noncommutative finite-dimensional division algebra. If D^* contains a locally nilpotent generating subgroup G, then D is a nilpotent crossed product.*

Proof: To prove this, it is enough to choose a basis for the vector space D

over $Z(D)$ from G, then consider the group generated by this basis and use Theorem 3.4. ∎

The multiplicative group of the real quaternion division algebra contains the quaternion group which is a generating subgroup. Therefore, by Theorem 3.4, it is cyclic. The following result says that if the multiplicative group of a noncommutative division algebra D contains a generating p-subgroup, then it is a nilpotent crossed product with $p = 2$ and $[D : F] = 2^m$ for some $m \in \mathbb{N}$.

COROLLARY 3.6: *Let D be a noncommutative finite-dimensional F-central division algebra. If D^* contains a generating p-subgroup, then D is a nilpotent crossed product with $[D : F] = 2^m$, for some $m \in \mathbb{N}$.*

Proof: Let G be a generating p-subgroup of D^*. Since G is locally nilpotent, by Corollary 3.5, we conclude that D is a nilpotent crossed product. If p is odd, then by a result of [3, p. 45], G is abelian, which contradicts the fact that G is a generating subgroup. If $p = 2$, then by the proof of Theorem 3.4, there exists a maximal subfield K of D such that $GK^*/K^* \simeq Gal(K/F)$ and K/F is a Galois extension. Hence $[K : F]$ is a power of 2 and the result follows. ∎

Let D be an F-central division algebra of prime degree. Suppose that D^* contains a generating soluble subgroup. Using the fact that the degree of D is prime, one may easily conclude that D^* contains a generating metabelian subgroup. Now, using Theorem 3.4, we obtain the following corollary, which is the main result of [1].

COROLLARY 3.7: *Let D be an F-central division algebra of prime degree p. Then D is cyclic if and only if D^* contains a nonabelian soluble subgroup.*

The following example shows that working with certain subgroups of D^* may be sometimes more useful than maximal subfields.

Example 1: Let L/K be a cyclic field extension of degree n, and denote by σ the generator of $Gal(L/K)$. Let $D = L((T, \sigma))$ be the division algebra of formal Laurent series. Although it is not hard to show that $L((T^n))/K((T^n))$ is a cyclic extension and therefore D is a cyclic division algebra, we would like to show its cyclicity by using our criterion. It is known that $Z(D) = K((T^n))$. If $1, t, \ldots, t^{n-1}$ is a basis for the field extension L/K, then $\{t^jT^i\}_{0\leq i,j<n}$ is a basis for D over $Z(D)$. Therefore, the group $G = \langle t^jT^i\rangle_{0\leq i,j<n}$ is a generating subgroup of D^*. On the other hand, one can easily show that $L \cap G \lhd G$ and that $G/L \cap G$ is a cyclic group. Therefore, G is a generating subgroup of D^*

which is abelian-by-cyclic. Thus, by Theorem 3.4, D is a cyclic division algebra. ∎

The rest of this section is devoted to the observation that one may not be able in general to replace "division algebra" by "central simple algebra" in the statements of some theorems above. First, we observe the following:

THEOREM 3.8: *Let D be an F-central crossed product division algebra; then for any natural number n, $GL_n(D)$ contains an abelian-by-finite generating subgroup.*

Proof: By Lemma 3.1, D^* contains an abelian-by-finite generating subgroup G, say. We may view each element of D as a diagonal matrix in $M_n(D)$. Suppose that M is the group of monomial matrices of $GL_n(F)$ containing $n \times n$ matrices with entries in F such that exactly one entry of each row and each column is nonzero. Let E be the group of diagonal matrices of $GL_n(F)$. Set $H = \langle G, M \rangle$. We claim that H as a subgroup of $GL_n(D)$ is a generating subgroup. One can easily show that $F[M] = M_n(F)$. Combining this equality with the fact that $F[G] = D$, we conclude that $F[H] = M_n(D)$. To complete the proof of the claim, it is enough to show that H is abelian-by-finite. It is easily seen that $M/E \simeq S_n$. Suppose that G/A is finite, where A is an abelian normal subgroup of G. Considering the fact that every element of G commutes with every element of M, we obtain $H = GM$ and that AE is an abelian group. Using these facts, one may easily show that H/AE as a homomorphic image of $G/A \times M/E$ is a finite group. Therefore, H is an abelian-by-finite generating subgroup of $GL_n(D)$, as desired. ∎

We close the paper with the following remark, which shows that in Theorem 3.4 we cannot replace "division algebra" by "central simple algebra".

Remark 2: Suppose that D is a crossed product division algebra and n a natural number. Using the notation of the proof of Theorem 3.8, we have $M/E \simeq S_n$. Therefore, M is an abelian-by-finite group. Hence, H is an abelian-by-finite generating subgroup of $GL_n(D)$. Thus, if D is an abelian crossed product (cyclic) division algebra, then $GL_2(D)$ contains a generating subgroup which is abelian-by-abelian (abelian-by-cyclic). Furthermore, if F is a field with more than 2 elements, by appealing to the group of monomial matrices M, we may find a generating subgroup of $GL_2(F)$ which is abelian-by-cyclic.

Example 2: We observe that there are some cyclic division algebras D, e.g., the division algebra of real quaternions $\mathbb{H}$, such that $M_n(\mathbb{H})$ is not a crossed product for any $n > 1$. In fact, it contains no subfield containing $\mathbb{R}$ which is of degree $2n$ over $\mathbb{R}$. But for each natural number n, $GL_n(\mathbb{H})$ contains an abelian-by-finite generating subgroup, and it also contains an abelian-by-cyclic generating subgroup for $n = 2$, but it is not a crossed product. Therefore, Theorem 3.4 is not valid if one replaces division algebras by central simple algebras.

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